

Memory in an N -qubit System as a Resource of the Decoherence Slow-Down with Application to Quantum Interferometry

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We demonstrate that memory in an N -qubit system subjected to decoherence, is a potential resource for the slow-down of the entanglement decay. As a possible application, we show that this effect can be used to retain the sub shot-noise sensitivity of the parameter estimation in quantum interferometry.

Introduction.— Non-classical correlations proved to be a key concept in many areas of modern science, such as quantum information [1], quantum computation, cryptography and teleportation [2] and even biology [3]. In ideal circumstances, non-classical states can be prepared and utilized at will. For instance in quantum metrology, correlated states can be employed to reach the Sub Shot-Noise (SSN) precision of parameter estimation [4–7]. However, a coupling with environment inevitably destroys subtle quantum correlations and drives the system into a classical mixture. This process, which is called *decoherence*, imposes severe limitations on what can be achieved in experiments [8–10]. The aim of this work is to study the impact of decoherence on the performance of two-mode atomic interferometers.

Along the line of recent interferometric experiments with ultra-cold atoms [11–15], we consider a collection of N bosonic qubits. Such system is equivalent to a single spin- $N/2$ particle and therefore can be described in terms of the algebra of angular momentum. For simplicity, we consider the interferometric protocol, which consists of an imprint of a phase θ between the two modes over a time τ [16]. In this work we take into account the coupling of the system to environment during the phase imprint. The impact of the decoherence on the preparation of the state has been studied in [17]. As our main result, we demonstrate – both analytically and numerically – that *memory* opens new possibilities to counteract the decoherence during the evolution of the two-mode quantum state under the influence of the environment.

Formulation of the problem.— A general approach for finding the dynamics of a system (S) interacting with the environment (E) relies upon solving the quantum Master equation [18, 19], $i\partial_t \hat{\rho}_S = \text{Tr}_E\{\hat{H}_{\text{tot}}, \hat{\rho}_{SE}\}$. Here we set $\hbar \equiv 1$ and $\hat{\rho}_{SE}$ is the density matrix of the system and the environment, while the reduced density matrix, $\hat{\rho}_S = \text{Tr}_E\{\hat{\rho}_{SE}\}$, is obtained by tracing out the environmental degrees of freedom. The Hamiltonian \hat{H}_{tot} is a sum of three parts \hat{H}_S , \hat{H}_E and \hat{H}_{int} . The Hamiltonian of the system is $\hat{H}_S = \Omega \hat{J}_z$, where Ω is an external field [20] related to the interferometric phase by $\theta = \Omega\tau$. Basing on recent experimental results [12, 13], we assumed that the two-body interactions, which are essential for the state preparation [4, 7], are absent during the phase imprint. The Hamiltonian of the environment – which couples with the system via \hat{H}_{int} – is \hat{H}_E .

The procedure described above is an *ab initio* method,

which in principle can always be employed to find the dynamics of the system. However, it is almost always impractical. First of all, construction of a realistic model of \hat{H}_E and \hat{H}_{int} is difficult because the necessary knowledge about the relation between numerous degrees of freedom is hardly accessible. Moreover, tracing out the environment is challenging and usually gives a closed equation for $\hat{\rho}_S$ only upon further radical approximations. The only known treatable case is the celebrated spin-boson model [19], which depicts a linear interaction of a single qubit with a collection of independent harmonic oscillators.

Below we discuss a phenomenological model of the system–environment interaction, which circumvents the difficulty with determining the detailed description of the environment. A necessary assumption for this construction is that the “back-action” of the system on the environment is either absent or can be safely neglected. If this is the case, the only way the system can “feel” the environment is through an effective external field $\Omega_E(t)$. Therefore, the interaction Hamiltonian is simply $\hat{H}_{\text{int}} = \Omega_E(t) \cdot \hat{\mathbf{J}}$, where $\hat{\mathbf{J}}$ is a vector of angular momentum operators. To determine the exact form of the field $\Omega_E(t)$ is as difficult as finding \hat{H}_E . Nevertheless, statistical properties of the environment should be much easier to either obtain or deduce. This observation allows for the final step, where we replace $\Omega_E(t)$ with a fluctuating field described by a stochastic process $\omega(t)$ chosen to reflect the aforementioned properties [21]. In this way, the explicit presence of the environmental degrees of freedom is mimicked by the fluctuations of the field and the total Hamiltonian \hat{H}_{tot} is replaced by

$$\hat{H} = \Omega \hat{J}_z + \omega(t) \cdot \hat{\mathbf{J}}. \quad (1)$$

The solution of the equations of motion provided by this Hamiltonian gives a density matrix $\hat{\rho}$, which is a stochastic process itself. It is necessary to average this result over the fluctuations – which is denoted by a bar – to obtain the density matrix of the system alone, $\hat{\rho}_S = \bar{\hat{\rho}}$. This step is a reminiscence of tracing over the environmental degrees freedom performed in the approach of the Master equation. In this way, we have constructed the general framework of the system–environment model. To continue further on it is necessary to specify the statistical properties of $\omega(t)$. Below we argue that the stationary Gaussian process is the most natural choice which accounts for the wide variety of physical situations.

Typically, we can expect that the environment consists of many approximately independent sources of fluctuations, which sum up to $\omega(t)$. If this is the case, then according to the Central Limit Theorem, $\omega(t)$ tends to a Gaussian process with the growing number of sources. Therefore, the process is fully determined by its average, $\overline{\omega(t)}$, and the correlation function, $\overline{\omega_i(t)\omega_j(t')} = \kappa_{ij}(t, t')$. Here, subscripts i, j denote the orthogonal components of the vector field $\omega(t)$. Moreover, the environment is expected to be in a stationary state – for example in the thermal equilibrium. In line with the “no-back-action” postulate which states that the system does not disturb the environment, there is no distinguished instant of time. Hence, also the process itself must be *stationary*, which implies that the average is a constant and the correlation function depends only on time difference. To make the discussion even more transparent, we choose $\omega(t)$ to be *isotropic* – which applies for an environment without any distinguished direction. To summarize,

$$\overline{\omega(t)} = 0 \quad \text{and} \quad \overline{\omega_i(t)\omega_j(t')} = \kappa(|t - t'|)\delta_{ij}, \quad (2)$$

where we set the average to be zero – without any loss of generality. In the majority of relevant situations, the correlation function $\kappa(|t - t'|)$ rapidly tends to zero when $|t - t'| > \tau_c$, where τ_c is the *correlation time*, which sets the time scale for the process [22]. The strength of the fluctuations is given by the variance of the process $\omega_0^2 = \kappa(0) = \overline{\omega_i^2(t)}$.

The non-zero correlation time connects the events from various instants of the evolution in a nontrivial way, i.e. not only through the initial conditions. Therefore, if the evolution is governed by a processes with $\tau_c \neq 0$, we say that the system has *memory*. Moreover, τ_c can be considered as a measure of memory, so the larger τ_c , the more memory in the system. In this work we demonstrate that presence of memory is a necessary ingredient for the slow-down of the entanglement decay. At this point, we are ready to employ the above model to determine the evolution of the density matrix of the system in presence of the noise.

Stochastic equations of motion.—The stochastic density matrix satisfies the von Neumann equation of motion $i\partial_t \hat{\rho}(t) = [\hat{H}, \hat{\rho}(t)]$, with the Hamiltonian given by Eq. (1). When $\hat{\rho}(t)$ is spanned by the appropriately chosen basis operators – which respect the symmetry of the Hamiltonian – the equations of motion greatly simplify. In our case this orthogonal basis is composed by a collection of *spherical tensor* operators $\hat{T}_m^{(j)}$ [23] and the density matrix reads

$$\hat{\rho}(t) = \sum_{j=0}^N \sum_{m=-j}^j \frac{\langle \hat{T}_m^{(j)}(t) \rangle^* \hat{T}_m^{(j)}}{\text{Tr} [\hat{T}_m^{(j)\dagger} \hat{T}_m^{(j)}]}, \quad (3)$$

where $\langle \hat{T}_m^{(j)}(t) \rangle = \text{Tr} [\hat{T}_m^{(j)}(t) \hat{\rho}(0)]$. The averaging over the fluctuations affects only $\langle \hat{T}_m^{(j)}(t) \rangle$ on the right-hand-side of Eq. (3), thus the problem of finding the evolution

of the system has been reduced to calculating these expectation values. Although this parametrization of the density matrix (3) might seem unnecessarily complicated, it gives a particularly simple set of equations

$$i\partial_t \langle \hat{T}_m^{(j)}(t) \rangle = \sum_{m'=-j}^j \mathbb{H}_{mm'}^{(j)}(t) \langle \hat{T}_{m'}^{(j)}(t) \rangle, \quad (4)$$

where $\mathbb{H}_{mm'}^{(j)}(t) = \langle j, m' | \hat{H} | j, m \rangle$ and $|j, m\rangle$ are the eigenstates of the angular momentum operators, which will be consequently used as the basis states throughout this work. The formal solution of Eq. (4) is given in terms of the time-ordered exponential which acts on the initial expectation values $\langle \hat{T}_m^{(j)} \rangle = \text{Tr} [\hat{T}_m^{(j)} \hat{\rho}(0)]$, i.e.

$$\langle \hat{T}_m^{(j)}(t) \rangle = \sum_{m'} \left[\mathcal{T} \exp \left(-i \int_0^t dt' \mathbb{H}^{(j)}(t') \right) \right]_{mm'} \langle \hat{T}_{m'}^{(j)} \rangle. \quad (5)$$

For Gaussian processes, this result can be averaged analytically using the “cummulant expansion” method described in detail in [24] and the outcome is

$$\overline{\langle \hat{T}_m^{(j)}(t) \rangle} = e^{-m^2 \Gamma_0 - (j(j+1) - m^2) \Gamma_+} e^{-im\varphi} \langle \hat{T}_m^{(j)} \rangle. \quad (6)$$

The decay rates are defined using a function $\Gamma(t, \Omega) = \int_0^t dt_1 \int_0^{t_1} dt_2 \kappa(t_2) e^{i\Omega t_2}$ and read $\Gamma_0 = \Gamma(t, 0)$, while $\Gamma_+ = \text{Re}[\Gamma(t, \Omega)]$. The phase factor is $\varphi = \Omega t - \Gamma_-$, where $\Gamma_- = \text{Im}[\Gamma(t, \Omega)]$. The two rates Γ_{\pm} result from the fluctuations in the $x - y$ plane, while Γ_0 originates only from the z -component of the noise.

Equations (3) and (6) provide a general scheme for finding the density matrix of the system evolving under the stochastic Hamiltonian (1). From Eq. (6) it is evident that the elements of the density matrix are damped by the fluctuations. It will be demonstrated below, that the rate of damping can be substantially decreased in presence of memory.

Evolution of the density matrix.—We illustrate the effect of memory by choosing $\hat{\rho}(0) = |\Psi\rangle\langle\Psi|$, where $|\Psi\rangle = \frac{1}{\sqrt{2}} (|\frac{N}{2}, \frac{N}{2}\rangle + |\frac{N}{2}, -\frac{N}{2}\rangle)$ is the maximally entangled “NOON” state. This state decomposed into the basis of spherical tensors in a following way

$$\hat{\rho}(0) = \sum_{j=0}^{\frac{N}{2}} \frac{\langle \hat{T}_0^{(2j)} \rangle^* \hat{T}_0^{(2j)}}{\text{Tr} [\hat{T}_0^{(2j)\dagger} \hat{T}_0^{(2j)}]} + \frac{\frac{1}{2} (\hat{T}_N^{(N)} + \hat{T}_{-N}^{(N)})}{\sqrt{\text{Tr} [\hat{T}_0^{(N)\dagger} \hat{T}_0^{(N)}]}}. \quad (7)$$

The first term is the diagonal of the density matrix spanned on tensors with $m = 0$. On the off-diagonal, the only non-zero elements are those in the top (spanned on the $m = N$ tensors) and the bottom ($m = -N$) corners. The presence of these tensors with $m \neq 0$ in the density matrix of the NOON state is crucial for the SSN sensitivity of the \hat{J}_z interferometer [16]. According to Eq. (6) the two contributions to the decay rate of the off-diagonal terms are the parameter-independent $N^2 \Gamma_0$ and

$N\Gamma_+$, which depends on Ω . The diagonal tensors decay only due to $j(j+1)\Gamma_+$. Asymptotically only the scalar $\hat{T}_0^{(0)} \propto \hat{1}$ remains thus the density matrix tends to a completely mixed state $\frac{1}{N+1}\hat{1}$ which does not depend on Ω and is useless for parameter estimation. This example confirms that the fluctuations indeed lead to the decoherence of the system. Although the decay into a completely mixed state at large times is inevitable, we will demonstrate below that long memory can significantly slow down this process.

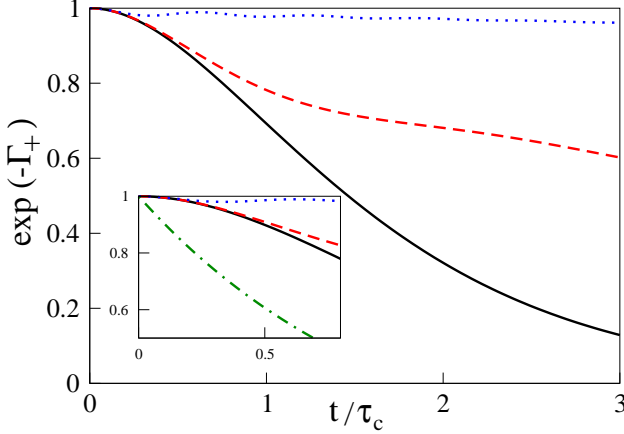


FIG. 1. (color online) The temporal behavior of the damping factor $\exp(-\Gamma_+)$ for $\Omega\tau_c = 0$ (solid black), $\Omega\tau_c = 2.5$ (dashed red) and $\Omega\tau_c = 10$ (dotted blue). The value of Γ_+ is calculated for the Ornstein-Uhlenbeck process [27], for which $\kappa(t) = \omega_0^2 \exp\left(-\frac{|t|}{\tau_c}\right)$ with $\omega_0\tau_c = 1$. The inset focuses on early times and compares the colored-noise cases with the white-noise limit $\exp(-\omega_0^2 t)$ (dot-dashed green).

In order to appreciate the role of memory as a resource for the decoherence slow-down, we compare the $\tau_c \neq 0$ case with the “white noise” limit, which is achieved by letting $\tau_c \rightarrow 0$ while keeping $\frac{1}{T} \equiv \int_0^\infty \kappa(s)ds$ constant. As a result, the correlation function tends to the Dirac delta, $\kappa(t) \rightarrow \frac{2}{T}\delta(t)$, which gives $\Gamma(t, \Omega) \rightarrow \frac{t}{T}$. On the other hand, the colored noise decay rate for short times is $\Gamma(t, \Omega) \simeq \frac{1}{2}\omega_0^2 t^2$. These two decay rates can be related by noting that $\int_0^\infty \kappa(s)ds \simeq \omega_0^2 \tau_c$, thus $\frac{1}{T}$ should be compared to the combination of the colored-noise parameters $\omega_0^2 \tau_c$. Therefore, according to Eq. (6) the white-noise decay is purely exponential, $e^{-j(j+1)\frac{\tau_c}{T}}$, while in the presence of memory and for $t \ll \tau_c$ the decay is less violent, namely $e^{-\frac{1}{2}j(j+1)\frac{\tau_c}{T}(\frac{t}{\tau_c})^2}$. As illustrated in the inset of Fig. 1, which shows the temporal behavior of the damping factor at short times, the memory unlocks a new time-scale for experiments at which the decoherence *build-up* is slow. In contrast, the memory-less white noise case is characterized by the omnipresent exponential decay of coherence.

Another way to control the decoherence results from an interplay between the deterministic field Ω and the fluctuating part $\omega(t)$. To picture this effect, we note that

every realization of $\omega(t)$ yields a realization of stochastic $\hat{\varrho}(t)$ as a solution of the von Neumann equation of motion. Each realization can be viewed in an approximate “stroboscopic” picture, where the field $\omega(t)$ does not change much within correlation time τ_c , and then jumps to another value. The aforementioned interplay becomes clear when the evolution is viewed from the reference frame rotating around Ω . Transition to this frame does not change the z component of the fluctuations which is manifested by the Ω -independent decay rate Γ_0 . On the other hand, the perpendicular part rotates around the z -axis with frequency Ω . If the period of the rotation is small in comparison to the correlation time ($\Omega\tau_c \gg 1$), it performs many revolutions between jumps. In such a case, stochastic jumps are not noticeable anymore, different realizations become indistinguishable thus one cannot speak of fluctuations anymore. This *gyroscopic effect* is exhibited by the decrease of Γ_+ , as shown in Fig. 1, which compares the damping factors $e^{-\Gamma_+}$ for three different values of $\Omega\tau_c$. Note that in the white-noise limit, the jumps are instantaneous which leaves no time for the gyroscopic effect to kick in and consequently the decay rate $\frac{1}{T}$ does not depend on Ω .

In the next step we examine how the build-up and gyroscopic effects can be utilized to preserve the SSN sensitivity in the interferometric parameter estimation.

Memory in noisy quantum interferometry—The precision of the estimation of the parameter Ω from a series of n experiments is bounded by the Cramer-Rao Lower Bound (CRLB) [25],

$$\Delta^2 \Omega \geq \frac{1}{n} \frac{1}{F_Q}, \quad (8)$$

where F_Q is called the Quantum Fisher Information (QFI), which depends on the state $\hat{\varrho}(\tau)$ used for estimation, namely

$$F_Q = 2 \sum_{i,j} \frac{|\langle i | \partial_\Omega \hat{\varrho}(\tau) | j \rangle|^2}{p_i + p_j}. \quad (9)$$

Here, $|i\rangle$ denotes the eigen-state of $\hat{\varrho}(\tau)$ with a corresponding eigen-value p_i . When $F_Q > N\tau^2$, the two-mode state $\hat{\varrho}(\tau)$ is usefully entangled from the interferometric point of view [5, 7] and via the CRLB (8) allows for the SSN sensitivity. In the absence of noise, it is the NOON state used in the previous section that provides the maximal estimation precision, as $F_Q = N^2\tau^2$ reaches the Heisenberg Limit (HL) [5, 7]. As argued above, when this state is exposed to noise for a long time, it reaches a completely mixed state, which gives $F_Q = 0$ and thus is utterly useless for interferometry. However, in presence of memory effects, so that $N\Gamma_+$ can be considered small, the N^2 scaling of the QFI which reads

$$F_Q \approx \frac{N^2}{1 - N\Gamma_+} |\tau - \partial_\Omega \Gamma_-|^2 e^{-2N^2\Gamma_0 - 2N\Gamma_+} \quad (10a)$$

$$+ N\Gamma_+ [\partial_\Omega \log \Gamma_+]^2 \quad (10b)$$

decays slower than in the white-noise limit

$$F_Q^{(\text{wn})} = N^2 \tau^2 e^{-2N(N+1)\frac{\tau}{T}}. \quad (11)$$

Moreover, additional information about Ω comes from the dependence of the decay rates on the parameter. We stress that this improvement is purely a memory effect, absent in Eq. (11). In both cases, the strongest source of decoherence – which cannot be diminished by the gyroscopic effect – comes from the z -component of the noise $\omega_z(t)$ which sets Γ_0 . Note that in the Hamiltonian (1),

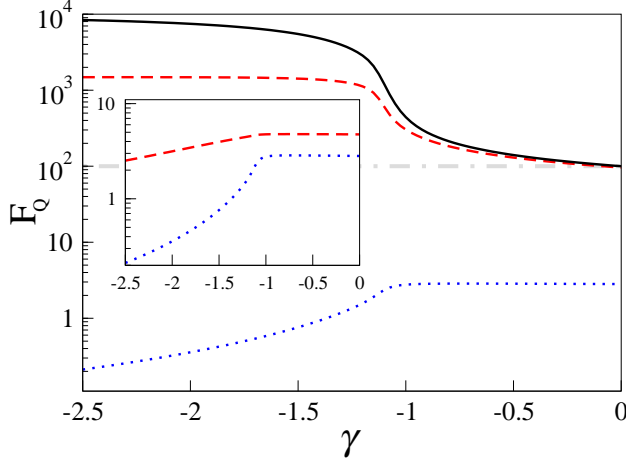


FIG. 2. (color online) The QFI in units of τ^2 for different input states $\hat{\rho}(0)$ of $N = 100$ particles parametrized with γ of the Bose-Hubbard Hamiltonian (see text for details) and with $\omega_z(t) = 0$. The Figure compares the QFI at $\tau = 0.5\tau_c$ in absence of noise (solid black), in presence of memory (dashed red) and in the white-noise limit (dotted blue). The grey horizontal dot-dashed line denotes the shot-noise limit $F_Q = 100$. The inset shows the QFI for the isotropic case in presence of memory (dashed red) and in the white-noise limit (dotted blue). As in Fig. 1 the colored noise is the Ornstein-Uhlenbeck process with $\omega_0\tau_c = 1$ and $\Omega\tau_c = 10$.

$\omega_z(t)$ can be interpreted as fluctuations of the estimated parameter itself. In this spirit we relax the isotropic assumption (2) and let the strength and correlation time of $\omega_z(t)$ be different from the perpendicular components. It is reasonable to assume that the experiment is set up in such a way, that the estimated parameter is well defined. That is, the fluctuations along the z -axis should be slow and weak. In particular, by setting $\omega_z = 0$, we get $\Gamma_0 = 0$ so the strongly damping exponential factor in line (10a) disappears. Also, in the white-noise limit the decay rate is less violent and equal to $2N/T$ rather than $2N(N+1)/T$ in Eq. (11).

To picture the impact of noise, in Fig. 2 we plot the QFI for a wide family of usefully entangled input states $\hat{\rho}(0)$. These states are the ground states of the two-well Bose-Hubbard Hamiltonian $\hat{H}_{\text{BH}} = -E_J \hat{J}_x + U \hat{J}_z^2$ with attractive two-body interactions [16, 26] for different values of the ratio $\gamma = U/NE_J$. And so, for $\gamma = 0$ we obtain a coherent state, while for $\gamma \rightarrow -\infty$ it tends to a NOON

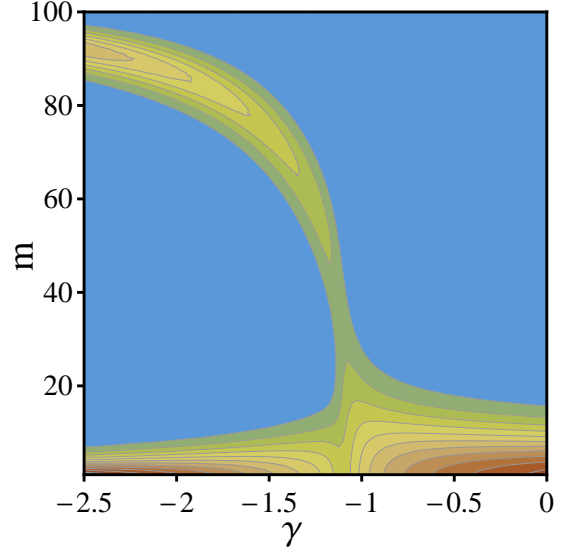


FIG. 3. (color online) Illustration of the structure of the initial density matrices of the states generated with the Bose-Hubbard Hamiltonian. The contour plot shows the distribution $\sum_j \left| \text{Tr} \left(\hat{\rho}(0) \hat{T}_m^{(j)} \right) \right|^2 / \text{Tr} \left(\hat{T}_m^{(j)\dagger} \hat{T}_m^{(j)} \right)$. For $\gamma \simeq 0$, the density matrix consist of close-to-diagonal terms, so only the tensors with $m \simeq 0$ contribute. When $\gamma \lesssim -1$, there is a growing contribution from large m 's.

state. The structure of this family of states in terms of the spherical tensors is presented in Fig. 3. Figure 2 shows that for $\omega_z(t) = 0$, memory effects prevent the significant information loss and keep the particles entangled, as $F_Q > N\tau^2$. To contrary, in the white-noise case, the SSN sensitivity is lost even at short times. Moreover, the inset underlines the destructive role of the fluctuations of the parameter Ω .

Conclusions and acknowledgements— We have demonstrated how the memory in an N -qubit system, which is a subject to external noise, can be employed to suppress the entanglement decay. We have argued that this effect has an application to quantum interferometry. In presence of memory, the deterministic evolution – which imprints the parameter- Ω dependence on the state – couples to the non-unitary evolution generated by the fluctuations. As a result, the SSN sensitivity is retained for longer times – not only due to the decoherence slow-down but also thanks to additional information about Ω present in the decay rates.

P. Sz. acknowledges the Foundation for Polish Science International Ph.D. Projects Program co-financed by the EU European Regional Development Fund. J. Ch. acknowledges the Foundation for Polish Science International TEAM Programme co-financed by the EU European Regional Development Fund. This work was supported by the National Science Center grant no. DEC-2011/03/D/ST2/00200. M. T. was supported by the National Science Center grant.

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- $$[\hat{J}_\pm, \hat{T}_m^{(j)}] = \sqrt{j(j+1) - m(m \pm 1)}\hat{T}_{m \pm 1}^{(j)},$$
- where $\hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y$.
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